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2001 J. Phys. A: Math. Gen. 34 5969

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Bound states in mildly curved layers

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Received 3 April 2001

Published 20 July 2001

Online at stacks.iop.org/JPhysA/34/5969

Abstract

It has been shown recently that a nonrelativistic quantum particle constrained to a hard-wall layer of constant width built over a geodesically complete simply connected noncompact curved surface can have bound states, provided the surface is not a plane. In this paper we study the weak-coupling asymptotics of these bound states, i.e., the situation when the surface is a mildly curved plane. Under suitable assumptions about regularity and decay of surface curvatures we derive the leading order in the ground-state eigenvalue expansion. The argument is based on Birman–Schwinger analysis of Schrödinger operators in a planar hard-wall layer.

PACS numbers: 03.65.Ge, 02.40.-k, 02.70.Hm, 03.65.Vf

1. Introduction

The investigation of quantum particles constrained to a spatial region Ω of a prescribed shape, in particular, relations between its spectral properties and the geometry of Ω , became an attractive problem when the progress in technology made it possible to fabricate various mesoscopic systems for which this is a reasonable model—see, e.g., [LCM]. Curvature induced bound states in hard-wall strips and tubes were demonstrated more than a decade ago [ES] and studied subsequently in numerous papers—cf [DE, LCM] and references therein.

Much less attention was paid to quantum mechanics in layers, apart from the trivial planar case. While for an experimenter it is easier to prepare a semiconductor film on a curved substrate than to fabricate a quantum wire, from the mathematical point of view the opposite is true, and solving the Schrödinger equation in a nontrivial layer is more complicated than the corresponding problem in a tube. It was noticed a long time ago that in the formal limit of zero width the layer curvatures give rise to an effective attractive potential [KJ] but the first results

for curved finite-width layers, with the particle Hamiltonian being a multiple of the Dirichlet Laplacian $-\Delta_D^\Omega$, appeared only recently.

We restrict ourselves to the case when Ω is noncompact and nontrivially curved. The first proof of the existence of geometrically induced bound states was given in [DEK1] under the assumption that the planar layer is curved only locally. A more general case of curved layers which are *asymptotically* planar, in the sense that the curvatures of the generating surface vanish at large distances, has been discussed in [DEK2]. There we have derived several sufficient conditions for the existence of bound states expressed in terms of geometric quantities characterizing the reference surface. While these conditions cover a wide class of layers, they do not represent an ultimate result: it is not clear, e.g., whether bound states exist in layers built over surfaces of positive total Gauss curvature unless the latter are thin enough or endowed with a cylindrical symmetry.

After demonstrating their existence one is naturally interested in the properties of the curvature-induced bound states. A particular question which we are going to address in the present paper concerns the weak-coupling regime. Since in our case the binding comes from the curvature alone, the situation described is expected to occur in *mildly curved* layers. We will show that the layer then has a unique eigenvalue and derive an asymptotic expansion for the gap between this eigenvalue and the threshold of the essential spectrum. The leading term in this formula will depend on the mean curvature of the reference surface.

Let us describe briefly the contents of this paper. In the next section we will give a precise formulation of the problem and describe the main result summarized in theorem 2.1. The rest is devoted to the proof. The strategy is adopted from the ‘lower-dimensional’ case of curved strips and tubes analysed in [DE, section 4]. First, in section 3, we consider Schrödinger operators acting in a planar layer built over \mathbb{R}^2 . We derive a necessary and sufficient condition under which such an operator has a bound state in the weak coupling limit and find the asymptotic expansion of the eigenvalue. The result contained in theorem 3.4 is of independent interest; we prove it in greater generality for a layer in \mathbb{R}^{2+m} , $m \geq 1$. Next, in section 4, we apply it to the case of mildly curved quantum layers. We express the operator $-\Delta_D^\Omega$ in the coordinates (x, u) , where $x \equiv (x^1, x^2) \in \mathbb{R}^2$ and $u \in I := (-a, a)$ with $a > 0$ parametrize the surface and its normal space, respectively, and pass to a unitarily equivalent operator with an effective potential, which can be consecutively estimated by operators to which theorem 3.4 can be applied. We will also show how the leading term in the expansion looks in the case of a thin layer.

2. The results

To give a precise statement of the main result we need first to specify what we mean by mildly curved quantum layers. Let a family of surfaces $\Sigma_\varepsilon := p(\mathbb{R}^2)$ be given by a Monge patch:

$$p : \mathbb{R}^2 \rightarrow \mathbb{R}^3 \quad p(x^1, x^2; \varepsilon) := (x^1, x^2, \varepsilon f(x^1, x^2)) \quad (2.1)$$

where f is supposed to be a C^4 -smooth function. Here $\varepsilon > 0$ is the parameter which controls the deformation; it is supposed to be small so that Σ_ε is a mildly curved plane. The cross-product of the tangent vectors $p_{,\mu} := \partial p / \partial x^\mu$, $\mu = 1, 2$, defines a unit normal field n on Σ_ε . We put $\Omega_0 := \mathbb{R}^2 \times I$ and define a layer $\Omega_\varepsilon := \mathcal{L}(\Omega_0)$ of width $d = 2a$ over the surface Σ_ε by virtue of the mapping $\mathcal{L} : \Omega_0 \rightarrow \mathbb{R}^3$ which acts as

$$\mathcal{L}(x, u; \varepsilon) := p(x; \varepsilon) + u n(x; \varepsilon). \quad (2.2)$$

The layer in question is thus the spatial region closed between two parallel surfaces—cf [Sp3, Prob. 12 of chapter 3]—represented by $p \pm an$. In this sense, we will hereafter refer to the surface coordinates (x^1, x^2) as longitudinal variables, while u will be the transverse variable.

We consider a nonrelativistic spinless particle confined to Ω_ε which is free within it, and suppose that the boundary of the layer is a hard wall, i.e. the wavefunctions satisfy the Dirichlet boundary condition there. For the sake of simplicity we set Planck's constant $\hbar = 1$ and the mass of the particle $m = \frac{1}{2}$. Then the Hamiltonian of the system can be identified with the Dirichlet Laplacian $-\Delta_D^{\Omega_\varepsilon}$ on $L^2(\Omega_\varepsilon)$, which is defined for an open set $\Omega_\varepsilon \subset \mathbb{R}^3$ as the Friedrichs extension of the Laplacian acting on $C_0^\infty(\Omega_\varepsilon)$ —cf [RS4, section XIII.15] or [Dav, chapter 6]. The domain of the closure of the corresponding quadratic form is the Sobolev space $W_0^{1,2}(\Omega_\varepsilon)$.

Next we must introduce some notation and formulate assumptions about the function f , in addition to the smoothness requirement mentioned above. We consider asymptotically planar surfaces—cf [DEK2]—which means that the Gauss K and mean M curvatures of Σ_ε vanish at large distances from a fixed point. This will hold if we require

$$\langle \text{d1} \rangle f_{,\mu} \in L^\infty(\mathbb{R}^2) \quad \text{together with} \quad \langle \text{d2} \rangle f_{,\mu\nu} \rightarrow 0 \text{ as } |x| \rightarrow \infty$$

for $\mu, \nu \in \{1, 2\}$. This allows us to localize the essential spectrum. Adapting from [DEK2, theorem 4.1] a simple argument based on a Neumann bracketing one finds that $\inf \sigma_{\text{ess}}(-\Delta_D^{\Omega_\varepsilon}) \geq \kappa_1^2$. Here $\{\kappa_j^2 := (\kappa_1 j)^2\}_{j=1}^\infty$, with $\kappa_1 := \pi/d$, are the eigenvalues of $-\Delta_D^I$; the corresponding eigenfunctions will be denoted by χ_j . Assuming in addition

$$\langle \text{d3} \rangle f_{,\mu\nu\rho} \rightarrow 0 \text{ as } |x| \rightarrow \infty \quad \text{and} \quad \langle \text{d4} \rangle f_{,\mu\nu\rho\sigma} \in L^\infty(\mathbb{R}^2)$$

for $\mu, \nu, \rho, \sigma \in \{1, 2\}$ we will be able to prove in section 4.3 that the bound is sharp:

$$\inf \sigma_{\text{ess}}(-\Delta_D^{\Omega_\varepsilon}) = \kappa_1^2. \tag{2.3}$$

We also have to require that the derivatives of f satisfy the following integrability hypotheses:

$$\langle \text{r1}, 2 \rangle f_{,\mu\nu}, f_{,\mu\nu\rho} \in L^2(\mathbb{R}^2, (1 + |x|^\delta) dx) \quad \text{respectively}$$

and

$$\langle \text{r3} \rangle f_{,\mu\nu\rho\sigma} \in L^1(\mathbb{R}^2, (1 + |x|^\delta) dx) \quad \text{for some } \delta > 0$$

and $\mu, \nu, \rho, \sigma \in \{1, 2\}$. The main result of the present work is as follows:

Theorem 2.1. *Let Ω_ε be a family of layers generated by the surfaces Σ_ε given by (2.1). Suppose that the function $f \in C^4(\mathbb{R}^2)$ satisfies the hypotheses $\langle \text{d1-4} \rangle$ and $\langle \text{r1-3} \rangle$. If Σ_1 is not planar, then for all ε small enough $-\Delta_D^{\Omega_\varepsilon}$ has exactly one isolated eigenvalue $E(\varepsilon)$ below the essential spectrum. Moreover, it can be expressed as*

$$E(\varepsilon) = \kappa_1^2 - e^{2w(\varepsilon)^{-1}}$$

where $w(\varepsilon)$ has the following asymptotic expansion:

$$w(\varepsilon) = -\varepsilon^2 \sum_{j=2}^\infty (\chi_1, u\chi_j)_I^2 (\kappa_j^2 - \kappa_1^2)^2 \int_{\mathbb{R}^2} \frac{|\hat{m}_0(\omega)|^2}{|\omega|^2 + \kappa_j^2 - \kappa_1^2} d\omega + \mathcal{O}(\varepsilon^{2+\gamma})$$

with $\gamma := \min\{1, \delta/2\}$. Here m_0 is the lowest-order term in the expansion of the mean curvature of Σ_ε w.r.t. ε —cf (4.3).

Remarks. (a) The subscript ‘ I ’ indicates the norm and the inner product in the space $L^2(I)$. The sum runs in fact over even n only because one integrates over the interval $I = (-a, a)$ on which the function $u \mapsto \chi_1(u)u\chi_j(u)$ is odd for odd j .

(b) The expression for the leading-term coefficient w_1 in the expansion $w(\varepsilon) =: \varepsilon^2 w_1 + \mathcal{O}(\varepsilon^{2+\gamma})$ does not have a very transparent structure. However, for thin layers it can be rewritten as

$$w_1 = -\frac{1}{2\pi} \|m_0\|^2 + \frac{\pi^2 - 6}{24\pi^3} \|\nabla m_0\|^2 d^2 + \mathcal{O}(d^4). \tag{2.4}$$

This formula is instructive because the first term comes from the surface attractive potential $K - M^2$ which dominates the picture in this case, while the $\mathcal{O}(d^4)$ error term expresses the contribution of higher transverse modes. We refer to section 4.4 for more details.

3. Weakly coupled Schrödinger operators in a planar layer

Let M be an open connected precompact set in \mathbb{R}^m , $m \geq 1$. The object of our interest in this section will be the operator

$$H_\lambda = -\Delta_D + \lambda V \quad \text{with } \lambda > 0 \quad \text{on } \mathcal{H} := L^2(\mathbb{R}^2) \otimes L^2(M) \quad (3.1)$$

where $-\Delta_D$ is the Dirichlet Laplacian on $\mathbb{R}^2 \times M$ defined as the closure of $-\Delta \otimes I_m + I_2 \otimes -\Delta_D^M$. In the last expression the unindexed $-\Delta$ stays for the Laplace operator in \mathbb{R}^2 . The potential V is supposed to be $H_0^{1/2}$ -bounded, in other words

$$\langle \mathbf{a1} \rangle \exists a, b \geq 0 \quad \forall \psi \in \text{Dom } Q_0 : \quad \|V\psi\| \leq a\|\psi\| + b\|H_0^{1/2}\psi\|$$

where $\text{Dom } Q_0 = W_0^{1,2}(\mathbb{R}^2 \times M)$ is the form domain of H_λ .

The Dirichlet Laplacian $-\Delta_D^M$ on $L^2(M)$ has a purely discrete spectrum consisting of eigenvalues $\kappa_1^2 < \kappa_2^2 \leq \dots \leq \kappa_j^2 < \dots$; the corresponding normalized eigenfunctions will be denoted by χ_j , where $j = 1, 2, \dots$. The lowest eigenvalue is, of course, simple and the eigenfunction χ_1 can be chosen positive—cf [RS4, section XIII. 12]. Important for our purposes are the transverse projections of the potential:

$$V_{jj'} := \int_M \bar{\chi}_j(y) V(\cdot, y) \chi_{j'}(y) \, dy$$

and the analogous quantities for other functions on $\mathbb{R}^2 \times M$, in particular for $|V|$. We shall adopt the assumptions

$$\langle \mathbf{a2} \rangle |V|_{11} \in L^{1+\delta}(\mathbb{R}^2) \quad \text{and} \quad \langle \mathbf{a3} \rangle |V|_{11} \in L^1(\mathbb{R}^2, (1 + |x|^\delta) \, dx)$$

for some $\delta > 0$. We also suppose that the essential spectrum of H_λ does not start below the lowest transverse-mode threshold, i.e.

$$\langle \mathbf{a0} \rangle \inf \sigma_{\text{ess}}(H_\lambda) \geq \kappa_1^2.$$

This is true, in particular, if V vanishes at large distances from a fixed point.

The goal of this section is to show that, for λ small enough, the discrete spectrum of H_λ below κ_1^2 is not empty, provided the projection of V onto χ_1 is not repulsive in the mean. This part of the spectrum then contains only one eigenvalue $E(\lambda)$ for such a small λ and it approaches κ_1^2 as λ tends to zero. The last claim follows from the following elementary fact.

Proposition 3.1. *Assume $\langle \mathbf{a1} \rangle$. Then there are positive constants λ_0 and c such that, for all $\lambda \in (0, \lambda_0)$, we have $H_\lambda \geq \kappa_1^2 - c\lambda$.*

Proof. Using the Schwarz inequality and the assumption $\langle \mathbf{a1} \rangle$ together with the inequality between the geometric and arithmetic means, we have for all $\psi \in \text{Dom } Q_0$ the bound

$$\begin{aligned} Q_\lambda[\psi] &:= \|H_\lambda^{1/2}\psi\|^2 \geq \|H_0^{1/2}\psi\|^2 - \lambda\|\psi\|\|V\psi\| \\ &\geq (1 - \lambda b/2) \|H_0^{1/2}\psi\|^2 - \lambda(a + b/2)\|\psi\|^2. \end{aligned}$$

Since $\|H_0^{1/2}\psi\| \geq \kappa_1\|\psi\|$, we obtain the assertion by putting $\lambda_0 := 2/b$ and $c := a + (1 + \kappa_1^2)b/2$. \square

3.1. Birman–Schwinger analysis

Using the orthonormal basis of $L^2(M)$ given by $\{\chi_j\}$, the free resolvent operator $R_0(\alpha) := (H_0 - \alpha^2)^{-1}$ is decoupled in the following way:

$$R_0(\alpha) = \sum_{j=1}^{\infty} \chi_j (-\Delta + k_j(\alpha)^2)^{-1} \bar{\chi}_j \quad k_j(\alpha) := \sqrt{\kappa_j^2 - \alpha^2}.$$

If we are interested in eigenvalues of H_λ below the lowest transverse mode, we have to consider $\alpha \in [0, \kappa_1)$ and the two-dimensional resolvent in the middle of the expansion is well defined for any $j = 1, 2, \dots$. It can be expressed in terms of Hankel’s function—cf [AGHH, chapter I.5]. Passing to Macdonald’s functions K_0 by [AS, 9.6.4], we arrive at the following integral kernel formula:

$$R_0(x, y, x', y'; \alpha) = \frac{1}{2\pi} \sum_{j=1}^{\infty} \chi_j(y) K_0(k_j(\alpha)|x - x'|) \bar{\chi}_j(y'). \tag{3.2}$$

Define $K(\alpha) := |V|^{1/2} R_0(\alpha) V^{1/2}$, where we have employed the usual sign convention, $V^{1/2} := |V|^{1/2} \text{sgn } V$. According to the Birman–Schwinger principle—cf [Sim]—the function $\alpha(\lambda)^2 \equiv E(\lambda)$ is an eigenvalue of H_λ if and only if the operator $\lambda K(\alpha)$ has the eigenvalue -1 , i.e.

$$\alpha^2 \in \sigma_{\text{disc}}(H_\lambda) \iff -1 \in \sigma_{\text{disc}}(\lambda K(\alpha)). \tag{3.3}$$

The first term in the expansion (3.2) referring to $j = 1$ has a singularity at $\alpha = \kappa_1$, and as usual in such problems we have to single it out. For this purpose we first introduce the following decomposition.

Lemma 3.2. *There are real-analytic functions f and g such that*

$$\forall u \in (0, \infty) : \quad K_0(u) = f(u) \ln u + g(u)$$

and

- (i) $f(u) = -1 + \mathcal{O}(u^2)$, $g(u) = (\ln 2 - \gamma_E) + \mathcal{O}(u^2)$ as $u \rightarrow 0+$,
- (ii) $\exists C_1 > 0 \forall u \in (0, \infty) : \max\{f(u), g(u)\} \leq C_1 e^{-u}$,

where γ_E denotes Euler’s constant.

Proof. Around the origin a good choice to approximate the Macdonald function K_0 would be $f := -I_0$, where I_0 is the other modified Bessel function [AS, 9.6] but it has a poor behaviour at large distances. Hence we use an interpolation, for instance

$$\begin{aligned} f(u) &:= -e^{-u^2} I_0(u) - (1 - e^{-u^2}) K_0(u) \\ g(u) &:= e^{-u^2} I_0(u) \ln u + [1 + (1 - e^{-u^2}) \ln u] K_0(u). \end{aligned}$$

Using the relation [AS, 9.6.13], we obtain the behaviour at the origin (i). On the other hand, it follows by [AS, 9.7.2] that f and g have a faster-than-exponential decay at infinity, which, together with (i), gives that there is a positive C_1 such that (ii) holds. \square

We will thus use the decomposition $K(\alpha) = L_\alpha + M_\alpha$, where

$$L_\alpha(x, y, x', y') := -\frac{1}{2\pi} |V(x, y)|^{1/2} \chi_1(y) \ln k_1(\alpha) \chi_1(y') V(x', y')^{1/2}$$

contains the singularity and the regular M_α splits into two parts again, $M_\alpha = A_\alpha + B_\alpha$. The operator B_α is given by the projection of the resolvent on higher transverse modes, i.e. $B_\alpha := |V|^{1/2} R_0^\perp(\alpha) V^{1/2}$ with

$$R_0^\perp(\alpha) := \sum_{j=2}^{\infty} \chi_j (-\Delta + k_j(\alpha)^2)^{-1} \bar{\chi}_j \tag{3.4}$$

and the kernel of the remaining term is therefore

$$A_\alpha(x, y, x', y') := \frac{1}{2\pi} |V(x, y)|^{1/2} \chi_1(y) (K_0(k_1(\alpha)|x - x'|) + \ln k_1(\alpha)) \chi_1(y') V(x', y')^{1/2}.$$

We note that M_α is, by definition, well defined for $\alpha = \kappa_1$. In particular

$$A_{\kappa_1}(x, y, x', y') = -\frac{1}{2\pi} |V(x, y)|^{1/2} \chi_1(y) \left(\gamma_E + \ln \frac{|x - x'|}{2} \right) \chi_1(y') V(x', y')^{1/2}.$$

Furthermore, we have the following lemma. Since its proof is purely technical, we postpone it to section 3.2 below.

Lemma 3.3. *Assume (a1–3). Then there are positive C_2, C_3 and C_4 such that*

- (i) $\forall \alpha \in [0, \kappa_1] : \|M_\alpha\| < C_2,$
- (ii) $\|M_\alpha - M_{\kappa_1}\| \leq C_3 \lambda^\gamma,$ with $\gamma := \min\{1, \delta/2\},$
- (iii) $\left\| \frac{dM_\alpha(w)}{dw} \right\| < C_4 |w|^{-1}$ for λ sufficiently small, where $w := (\ln k_1(\alpha))^{-1}.$

We recall that proposition 3.1 yields $\alpha^2 \rightarrow \kappa_1^2 -$ as $\lambda \rightarrow 0+$, and consequently, $k_1(\alpha) \rightarrow 0+$. Hence the auxiliary variable w is well defined and negative for λ small enough, and $w \rightarrow 0-$ as $\lambda \rightarrow 0+$.

By the Birman–Schwinger principle (3.3) eigenvalues of H_λ correspond to singularities of the operator $(I + \lambda K(\alpha))^{-1}$ which we can equivalently express as

$$(I + \lambda K(\alpha))^{-1} = [I + \lambda(I + \lambda M_\alpha)^{-1} L_\alpha]^{-1} (I + \lambda M_\alpha)^{-1}.$$

Since M_α is bounded independently of α due to lemma 3.3(i), we have $\|\lambda M_\alpha\| < 1$ for all sufficiently small λ , and therefore the second term on the right-hand side of the last relation is a bounded operator. On the other hand, $\lambda(I + \lambda M_\alpha)^{-1} L_\alpha$ is a rank-one operator of the form $(\psi, \cdot)\varphi$, where

$$\begin{aligned} \psi(x, y) &:= -\frac{\lambda}{2\pi} \ln k_1(\alpha) V(x, y)^{1/2} \chi_1(y) \\ \varphi(x, y) &:= [(I + \lambda M_\alpha)^{-1} |V|^{1/2} \chi_1](x, y) \end{aligned}$$

so it has just one eigenvalue which is (ψ, φ) . The requirement that the latter equals -1 yields the implicit equation

$$w = F(\lambda, w) \quad F(\lambda, w) := \frac{\lambda}{2\pi} (V^{1/2} \chi_1, (I + \lambda M_{\alpha(w)})^{-1} |V|^{1/2} \chi_1) \quad (3.5)$$

where we use the auxiliary variable w defined in lemma 3.3 which determines the energy via $\alpha^2 = \kappa_1^2 - e^{2w^{-1}}$. Solving (3.5), we arrive at the main result of this section.

Theorem 3.4. *Assume (a0–3) and exclude the trivial case, $V \equiv 0$. Then H_λ , for sufficiently small $\lambda > 0$, has exactly one eigenvalue $E(\lambda)$ if and only if*

$$\int_{\mathbb{R}^2} V_{11}(x) dx \leq 0 \quad (3.6)$$

and in this case we can express it as $E(\lambda) = \kappa_1^2 - e^{2w(\lambda)^{-1}}$, where $w(\lambda)$ has the following asymptotic expansion:

$$\begin{aligned} w(\lambda) &= \frac{\lambda}{2\pi} \int_{\mathbb{R}^2} V_{11}(x) dx + \left(\frac{\lambda}{2\pi} \right)^2 \left\{ \int_{\mathbb{R}^2 \times \mathbb{R}^2} V_{11}(x) \left(\gamma_E + \ln \frac{|x - x'|}{2} \right) V_{11}(x') dx dx' \right. \\ &\quad \left. - \sum_{j=2}^{\infty} \int_{\mathbb{R}^2 \times \mathbb{R}^2} V_{1j}(x) K_0(k_j(\kappa_1)|x - x'|) V_{j1}(x') dx dx' \right\} + \mathcal{O}(\lambda^{2+\gamma}) \quad (3.7) \end{aligned}$$

with $\gamma := \min\{1, \delta/2\}.$

Proof. Inserting the identity

$$(I + \lambda M_\alpha)^{-1} = I - \lambda M_{\kappa_1} - \lambda(M_\alpha - M_{\kappa_1}) + \lambda^2 M_\alpha^2 (I + \lambda M_\alpha)^{-1}$$

into (3.5) and employing lemmas 3.3(i) and (ii), we get the asymptotic expansion (3.7). Note that its coefficients are well defined owing to (a3); in particular, for the second-order terms it is true since $(V^{1/2}\chi_1, M_{\kappa_1}|V|^{1/2}\chi_1)$ estimated by the Schwarz inequality is finite because of (a3) and lemma 3.3(i). The sufficient and necessary condition (3.6) follows from the fact that $E(\lambda)$ converges to κ_1^2 as $\lambda \rightarrow 0+$ because of proposition 3.1, which corresponds to the situation that w goes to zero assuming *negative* values. In view of (3.7) it is evident that this is the case if $\int V_{11}$ is strictly negative. We want to show that w is negative also for small λ if the first term in the expansion vanishes. Suppose first that the potential projections $V_{jj'}$ belong to $L^2(\mathbb{R}^2)$. Then we have

$$\begin{aligned} & \int_{\mathbb{R}^2 \times \mathbb{R}^2} V_{1j}(x) K_0(k_j(\kappa_1)|x - x'|) V_{j1}(x') dx dx' \\ &= (V_{1j}, K_0 * V_{j1}) = (\widehat{V}_{1j}, \widehat{K_0 * V_{j1}}) \\ &= 2\pi (\widehat{V}_{1j}, \widehat{K_0} \widehat{V}_{j1}) = (2\pi)^2 \int_{\mathbb{R}^2} \frac{|\widehat{V}_{1j}(\omega)|^2}{|\omega|^2 + \kappa_j^2 - \kappa_1^2} d\omega > 0 \end{aligned} \tag{3.8}$$

because $(2\pi)^{-1}K_0(k_j(\kappa_1)| \cdot |)$ is the Green function of $-\Delta + k_j(\kappa_1)^2$. At the same time, by lemma 3.2(i) and the fact that we deal with the case $\int V_{00} = 0$ now,

$$\begin{aligned} & - \int_{\mathbb{R}^2 \times \mathbb{R}^2} V_{11}(x) \left(\gamma_E + \ln \frac{|x - x'|}{2} \right) V_{11}(x') dx dx' \\ &= \lim_{\varepsilon \rightarrow 0+} \int_{\mathbb{R}^2 \times \mathbb{R}^2} V_{11}(x) (K_0(\varepsilon|x - x'|) + \ln \varepsilon) V_{11}(x') dx dx' \\ &= \lim_{\varepsilon \rightarrow 0+} \int_{\mathbb{R}^2 \times \mathbb{R}^2} V_{11}(x) K_0(\varepsilon|x - x'|) V_{11}(x') dx dx' \end{aligned} \tag{3.9}$$

is also positive, which follows by the consecutive use of the Fourier transform trick (3.8). For a general $V_{jj'} \notin L^2(\mathbb{R}^2)$ we can approximate the potential by the cut-off functions $V^N := \chi_{[-N, N]^2} \text{sgn} V \min\{|V|, N\}$, where $\chi_{\mathcal{A}}$ denotes the characteristic function of a set \mathcal{A} . Then the expressions in the first line of (3.8) and the last line of (3.9) are approximated by sequences whose elements are positive by the above argument. Using the dominated convergence and the absolute continuity of the Lebesgue integral we find that the limits exist, and of course they are positive again. Together we get that the second-order term in the expansion (3.7) is negative.

It remains to check that (3.7) is the only solution of (3.5) for λ small. This will be true if we prove it for a nonpositive V since it represents a more attractive interaction; we thus replace V by $-|V|$ in the expression for F . Using the Schwarz and triangle inequalities together with lemmas 3.3(i) and (iii), we arrive at the estimate

$$\begin{aligned} \left| \frac{\partial F}{\partial w}(\lambda, w) \right| &= \frac{\lambda}{2\pi} \left| \left(|V|^{1/2}\chi_1, (I + \lambda M_{\alpha(w)})^{-1} \lambda \frac{dM_{\alpha(w)}}{dw} (I + \lambda M_{\alpha(w)})^{-1} |V|^{1/2}\chi_1 \right) \right| \\ &\leq \frac{\lambda^2}{2\pi} \| |V|^{1/2}\chi_1 \|^2 \| (I + \lambda M_{\alpha(w)})^{-1} \|^2 \left\| \frac{dM_{\alpha(w)}}{dw} \right\| \\ &\leq \frac{\lambda^2}{2\pi} (1 - \lambda C_2)^{-2} \frac{C_4}{|w|} \| |V|_{11} \|^2 \end{aligned}$$

which holds for λ sufficiently small and strictly less than C_2^{-1} . The norm of the potential is finite by assumption (a3). Excluding the trivial case $V \equiv 0$ we note that there exists

a $c' > 0$ such that the inequality $|w|^{-1} \leq c'\lambda^{-1}$ is valid for any solution w of the implicit equation (3.5) and λ small enough. So there is a $C_5 > 0$ such that the partial derivative of F w.r.t. w is bounded by $C_5\lambda$ for all sufficiently small λ . Since any two solutions w_1, w_2 of the equation $w = F(\lambda, w)$ have to fulfil

$$|w_2 - w_1| = \left| \int_{w_1}^{w_2} \frac{\partial F}{\partial w} dw \right| \leq \left| \int_{w_1}^{w_2} \left| \frac{\partial F}{\partial w} \right| dw \right| \leq C_5\lambda |w_2 - w_1|$$

the uniqueness is ensured for $\lambda < C_5^{-1}$. \square

3.2. Proof of lemma 3.3

A reader not interested in the following technical analysis of the operator M_α may skip this subsection. We recall that M_α is given by the sum of A_α and B_α , which are of a different nature. To prove the assertions of lemma 3.3 we consider each of the operators separately.

3.2.1. Analysis of A_α . We show first that this operator is of the Hilbert–Schmidt class for $\alpha = \kappa_1$.

Lemma 3.5. *Assume (a2), (a3), then A_{κ_1} is a Hilbert–Schmidt operator.*

Proof. Let us compute the HS-norm:

$$\|A_{\kappa_1}\|_{\text{HS}}^2 = \frac{1}{4\pi^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |V|_{11}(x) \left| \gamma_E + \ln \frac{|x-x'|}{2} \right|^2 |V|_{11}(x') dx dx'$$

It can be estimated by a sum of two integrals. The first one will be finite if $|V|_{11} \in L^1(\mathbb{R}^2)$ which is true by (a3), so it remains to check that the integral

$$J := \int_{\mathbb{R}^2 \times \mathbb{R}^2} |V|_{11}(x) \ln^2 |x-x'| |V|_{11}(x') dx dx'$$

is finite. To estimate it, we divide the region of integration into two parts: if $|x-x'| \geq 1$, then for any $\delta > 0$ there is a $C_\delta > 0$ such that

$$\ln^2 |x-x'| \leq \ln^2(|x|+|x'|) \leq \ln^2(1+|x|)(1+|x'|) \leq C_\delta(1+|x|^\delta)(1+|x'|^\delta)$$

and the contribution to J is thus finite because of (a3). On the other hand, for $|x-x'| < 1$ we use the Hölder and Young inequalities:

$$\begin{aligned} J &= \int_{\mathbb{R}^2} dx |V|_{11}(x) \int_{\mathbb{R}^2} dx' \chi_{[0,1]}(|x-x'|) \ln^2 |x-x'| |V|_{11}(x') \\ &\leq \| |V|_{11} \|_{1+\delta} \| \chi_{[0,1]} \ln^2 * |V|_{11} \|_{1+\delta^{-1}} \\ &\leq \| |V|_{11} \|_{1+\delta}^2 \| \chi_{[0,1]} \ln \|_r = \Gamma(r+1)^{1/r} \| |V|_{11} \|_{1+\delta}^2 \end{aligned}$$

where $r := (1+\delta^{-1})/2$, which yields a finite value owing to (a2). \square

In a similar way we can estimate the HS-norm of $A_\alpha - A_{\kappa_1}$.

Lemma 3.6. *Assume (a3), then $\|A_\alpha - A_{\kappa_1}\|_{\text{HS}} \leq C_6 k_1(\alpha)^\delta$ holds with a positive constant C_6 independent of α .*

Proof. Using the decomposition of lemma 3.2,

$$\begin{aligned} (A_\alpha - A_{\kappa_1})(x, y, x', y') &= |V(x, y)|^{1/2} \chi_1(y) \{ [f(k_1(\alpha)|x-x'|) - f(0)] \ln k_1(\alpha)|x-x'| \\ &\quad + g(k_1(\alpha)|x-x'|) - g(0) \} \chi_1(y') V(x', y')^{1/2} \end{aligned}$$

which yields the estimate $\|A_\alpha - A_{\kappa_1}\|_{\text{HS}}^2 \leq 2(J_1^2 + J_2^2)$, where

$$J_\ell := \int_{\mathbb{R}^2 \times \mathbb{R}^2} |V|_{11}(x) h_\ell(k_1(\alpha)|x - x'|) |V|_{11}(x') \, dx \, dx' \quad \ell = 1, 2$$

with $h_1(u) := [(f(u) - f(0)) \ln u]^2$ and $h_2(u) := [g(u) - g(0)]^2$. Since these functions are bounded, both integrals are finite under the assumption $|V|_{11} \in L^1(\mathbb{R}^2)$. Consequently, $\|A_\alpha - A_{\kappa_1}\|_{\text{HS}}$ has a bound independent of α . However, we want to prove in addition that A_α converges in HS-norm to A_{κ_1} , i.e. the stated assertion. By virtue of lemma 3.2, we have for any $\delta \in (0, 2)$ the rough bounds $h_\ell(u) \leq c_\ell u^\delta$ for some constants $c_\ell > 0$. This bound, together with $|x - x'|^\delta \leq \max\{1, 2^{\delta-1}\}(|x|^\delta + |x'|^\delta)$ and $|x|^\delta + |x'|^\delta \leq (1 + |x|^\delta)(1 + |x'|^\delta)$, yields the estimate

$$J_\ell \leq \tilde{C}_6 k_1(\alpha)^\delta \left(\int_{\mathbb{R}^2} |V|_{11}(x) (1 + |x|^\delta) \, dx \right)^2$$

where $\tilde{C}_6 := \max\{1, 2^{\delta-1}\} \max\{c_1, c_2\}$. Since the integral is finite by (a3), we arrive at the sought-after result. \square

It is now easy to check that the two preceding lemmas imply the claims (i) and (ii) of lemma 3.3 for the operator A_α , since $\|\cdot\| \leq \|\cdot\|_{\text{HS}}$ and $k_1(\alpha) = \mathcal{O}(\lambda^{1/2})$ as it follows from proposition 3.1.

It is also easy to see that $\alpha \mapsto A_\alpha$ considered as the operator-valued function is real-analytic in $[0, \kappa_1)$. It cannot be analytically continued to an open interval containing κ_1 . However, if we consider instead the function $w \mapsto A_{\alpha(w)}$ with the auxiliary variable $w = (\ln k_1(\alpha))^{-1}$, this one can be continued to a complex region that includes $w = 0$, which is the point of our interest because it is obtained in the limit $\lambda \rightarrow 0$. We use this fact to estimate the norm of the derivative w.r.t. w , which establishes claim (iii) of lemma 3.3 for the operator $A_{\alpha(w)}$.

Lemma 3.7. *Assume (a2) and (a3), then for sufficiently small w :*

$$\left\| \frac{dA_{\alpha(w)}}{dw} \right\| < \frac{C_7}{|w|} \quad \text{with some } C_7 > 0.$$

Proof. Let us choose the contour $\Gamma := \{z = w + we^{it} : t \in [0, 2\pi)\}$ in the complex plane, where w is supposed to be negative and small enough so that $A_{\alpha(w)}$ is analytic in the interior of the curve. We can use the Cauchy integral formula to estimate the complex derivative

$$\left\| \frac{dA_{\alpha(w)}}{dw} \right\| = \left\| \frac{1}{2\pi i} \oint_\Gamma \frac{A_{\alpha(z)}}{(z - w)^2} \, dz \right\| \leq \frac{\sup_{z \in \Gamma} \|A_{\alpha(z)}\|}{|w|}.$$

It is straightforward to modify the proof of lemma 3.6 (including the technical lemma 3.2) in order to check the continuity of A_α in κ_1 w.r.t. to the HS-norm also for complex values of α . This yields the desired result. \square

3.2.2. Analysis of B_α . Mimicking [BGRS, lemma 2.2] and [BEGK, lemma 2.3] we denote as $\mathcal{H}_1 \subset \mathcal{H}$ the subspace $L^2(\mathbb{R}^2) \otimes \{\chi_1\}$. Let \mathcal{P}_1 be the corresponding projection and $\mathcal{P}_1^\perp := I - \mathcal{P}_1$, then we can write $R_0^\perp(\alpha)$ defined in (3.4) as $\mathcal{P}_1^\perp R_0(\alpha) \mathcal{P}_1^\perp$.

Lemma 3.8. *Assume (a1), then $\alpha \mapsto B_\alpha$ is uniformly bounded in the operator-norm topology on the interval $[0, \kappa_2 - \varepsilon]$ for any $\varepsilon \in (0, \kappa_2)$.*

Proof. Since the lowest point in the spectrum of $H_0 \mathcal{P}_1^\perp \upharpoonright \mathcal{P}_1^\perp \mathcal{H}$ is κ_2^2 , the operator-valued function $R_0^\perp(\cdot)$ has an analytic continuation into the region $\{\alpha \in \mathbb{C} \mid \alpha^2 \in \mathbb{C} \setminus [\kappa_2^2, \infty)\}$.

In particular, this region includes the interval $[0, \kappa_1]$ actually considered, where one has the following estimate on the norm:

$$\|R_0^\perp(\alpha)\| = \sup_{j=2,3,\dots} (\kappa_j^2 - \alpha^2)^{-1} = (\kappa_2^2 - \alpha^2)^{-1} < (\kappa_2^2 - \kappa_1^2)^{-1}.$$

If V was essentially bounded then this result would be sufficient for the boundedness of B_α because $\|B_\alpha\| \leq \|V^{1/2}\|_\infty \|R_0^\perp(\alpha)\| \|V^{1/2}\|_\infty$. In order to accommodate the extra factors $|V|^{1/2}$, $V^{1/2}$ under our weaker assumption (a1), we introduce the quadratic form

$$b_\alpha(\phi, \psi) := (\phi, B_\alpha \psi) = (R_0^\perp(\alpha)^{1/2} \mathcal{P}_1^\perp |V|^{1/2} \phi, R_0^\perp(\alpha)^{1/2} \mathcal{P}_1^\perp V^{1/2} \psi)$$

with α supposed to be a real number from an interval $[0, \kappa_2 - \varepsilon]$. To check boundedness of this form, it is sufficient to verify that $R_0^\perp(\alpha)^{1/2} \mathcal{P}_1^\perp |V|^{1/2}$ is a bounded operator, i.e. that $|V|^{1/2}$ is $(R_0^\perp(\alpha)^{-1/2} \mathcal{P}_1^\perp)$ -bounded, which is equivalent to the statement that there exist $c_1, c_2 \geq 0$ such that

$$\forall \psi \in \text{Dom } Q_0 : \quad \|V^{1/2} \mathcal{P}_1^\perp \psi\|^2 \leq c_1 \|\psi\|^2 + c_2 \|R_0^\perp(\alpha)^{-1/2} \mathcal{P}_1^\perp \psi\|^2.$$

However

$$\begin{aligned} \|V^{1/2} \mathcal{P}_1^\perp \psi\|^2 &= (\mathcal{P}_1^\perp \psi, |V| \mathcal{P}_1^\perp \psi) \leq \|\mathcal{P}_1^\perp \psi\| \|V \mathcal{P}_1^\perp \psi\| \\ &\leq \|\mathcal{P}_1^\perp \psi\| (a \|\mathcal{P}_1^\perp \psi\| + b \|H_0^{1/2} \mathcal{P}_1^\perp \psi\|) \\ &\leq \left(a + \frac{b}{2}\right) \|\psi\|^2 + \frac{b}{2} \|H_0^{1/2} \mathcal{P}_1^\perp \psi\|^2. \end{aligned}$$

In the first step we have used the Schwarz inequality, in the second our hypothesis (a1), and finally the inequality between the geometric and arithmetic means together with $\|\mathcal{P}_1^\perp \psi\| \leq \|\psi\|$. At the same time,

$$\|R_0^\perp(\alpha)^{-1/2} \mathcal{P}_1^\perp \psi\|^2 = (\mathcal{P}_1^\perp \psi, (H_0 - \alpha^2) \mathcal{P}_1^\perp \psi) \geq \|H_0^{1/2} \mathcal{P}_1^\perp \psi\|^2$$

so we can identify $c_1 := a + b/2$ and $c_2 := b/2$, which completes our proof. \square

This establishes lemma 3.3(i) for the operator B_α . Since $k_1(\alpha)^2 \leq c\lambda$ owing to proposition 3.1, property (ii) is included in the following result.

Lemma 3.9. *Assume (a1), then there exists $C_7 > 0$ such that*

$$\|B_\alpha - B_{\kappa_1}\| \leq C_7 k_1(\alpha)^2.$$

Proof. Using the first resolvent identity [Wei, theorem 5.13], we infer

$$\begin{aligned} \|B_\alpha - B_{\kappa_1}\| &= |\alpha - \kappa_1| \| |V|^{1/2} R_0^\perp(\alpha) R_0^\perp(\kappa_1) V^{1/2} \| \\ &\leq |\alpha - \kappa_1| \| |V|^{1/2} R_0^\perp(\alpha)^{1/2} \| \|R_0^\perp(\alpha)^{1/2}\| \|R_0^\perp(\kappa_1)^{1/2}\| \|R_0^\perp(\kappa_1)^{1/2} V^{1/2}\|. \end{aligned}$$

However, $|\kappa_1 - \alpha| \leq \kappa_1^{-1} k_1(\alpha)^2$ and it is clear from the proof of lemma 3.8 that the remaining factors at the rhs of the above estimate are finite. \square

Similarly to the operator A_α , we also need a bound on the derivative of $B_{\alpha(w)}$ w.r.t. w . However, the situation in the present case is simpler because $R_0^\perp(\alpha)$ itself is analytic in an open complex set containing $\alpha = \kappa_1$.

Lemma 3.10. *Assume (a1), then for sufficiently small w ,*

$$\left\| \frac{dB_{\alpha(w)}}{dw} \right\| \leq C_8 \quad \text{with some } C_8 > 0.$$

Proof. Since

$$\frac{dB_{\alpha(w)}}{dw} = \frac{dB_{\alpha}}{d\alpha} \frac{d\alpha}{dw} = 2 \frac{e^{2w-1}}{w^2} |V|^{1/2} R_0^{\perp}(\alpha(w))^2 V^{1/2}$$

and the prefactor function $w^{-2}e^{2w-1}$ is uniformly bounded in $(-\infty, 0)$, we can employ the Schwarz inequality to get the estimate

$$\| |V|^{1/2} R_0^{\perp}(\alpha)^2 V^{1/2} \| \leq \| |V|^{1/2} R_0^{\perp}(\alpha)^{1/2} \| \| R_0^{\perp}(\alpha) \| \| R_0^{\perp}(\alpha) V^{1/2} \|.$$

It is clear from the proof of lemma 3.8 that the right-hand side of this inequality is uniformly bounded in $[0, \kappa_2 - \varepsilon]$ for any $\varepsilon \in (0, \kappa_2]$. The former interval contains a neighbourhood of κ_1 in which w is well defined. \square

This yields the remaining assertion (iii) of lemma 3.3 for $|w| < 1$.

4. Mildly curved layers

Our strategy in proving the ground-state asymptotic expansion in mildly curved layers Ω_{ε} will be to estimate the corresponding Hamiltonian $-\Delta_{\mathbb{D}}^{\Omega_{\varepsilon}}$ by an operator of the form $-\Delta - \Delta_{\mathbb{D}}^I + \varepsilon V$ and to apply theorem 3.4 to the latter. Here $-\Delta$ is the Laplacian in the plane, $-\Delta_{\mathbb{D}}^I$ is the transverse operator which is the particular case of $-\Delta_{\mathbb{D}}^M$ discussed in the preceding section, and V is an effective potential given by curvatures of Σ_{ε} .

4.1. The geometry

The family of metric tensors for the surfaces given by (2.1) has the form

$$g_{\mu\nu}(\varepsilon) = \delta_{\mu\nu} + \varepsilon^2 \eta_{\mu\nu} \quad \text{with} \quad (\eta_{\mu\nu}) := \begin{pmatrix} f_{,1}^2 & f_{,1} f_{,2} \\ f_{,1} f_{,2} & f_{,2}^2 \end{pmatrix}$$

where the symbol $\delta_{\mu\nu}$ (as well as $\delta^{\mu\nu}$) has to be understood as the identity matrix. Since $\det(\eta_{\mu\nu}) = 0$ we get immediately

$$g(\varepsilon) := \det(g_{\mu\nu}) = 1 + \varepsilon^2 \text{tr}(\eta_{\mu\nu}) = 1 + \varepsilon^2 (f_{,1}^2 + f_{,2}^2) \tag{4.1}$$

together with the expression for the inverse matrix:

$$g^{\mu\nu}(\varepsilon) = g(\varepsilon)^{-1} (\delta^{\mu\nu} + \varepsilon^2 \tilde{\eta}^{\mu\nu}) \quad \text{with} \quad (\tilde{\eta}^{\mu\nu}) := \begin{pmatrix} f_{,2}^2 & -f_{,1} f_{,2} \\ -f_{,1} f_{,2} & f_{,1}^2 \end{pmatrix}.$$

In particular, the Jacobian $g^{1/2}$ defines through $d\Sigma_{\varepsilon} := g^{1/2} dx$ the invariant surface element. We will suppose that the matrix function $\eta_{\mu\nu}$ is bounded. Since its eigenvalues are 0 and $f_{,1}^2 + f_{,2}^2$, this will be true provided the second eigenvalue is a bounded function in \mathbb{R}^2 , i.e. if we adopt the assumption $\langle d1 \rangle$. Denote the bound of $\eta_{\mu\nu}$ by η_{∞} . Then it follows that $g_{\mu\nu}(\varepsilon)$ is uniformly elliptic for sufficiently small ε because

$$c_- \delta_{\mu\nu} \leq g_{\mu\nu}(\varepsilon) \leq c_+ \delta_{\mu\nu} \quad \text{with} \quad c_{\pm} := 1 \pm \varepsilon^2 \eta_{\infty}. \tag{4.2}$$

At the same time, the vector $n(\varepsilon) = g(\varepsilon)^{-1/2} (-\varepsilon f_{,1}, -\varepsilon f_{,2}, 1)$ represents the surface normal and therefore the second fundamental form is given by

$$h_{\mu\nu}(\varepsilon) = \varepsilon g(\varepsilon)^{-1/2} \theta_{\mu\nu} \quad \text{with} \quad (\theta_{\mu\nu}) := \begin{pmatrix} f_{,11} & f_{,12} \\ f_{,21} & f_{,22} \end{pmatrix}.$$

We can now construct the Weingarten tensor [Kli, definition 3.3.4 and proposition 3.5.5]:

$$h_{\mu}^{\nu}(\varepsilon) := h_{\mu\rho} g^{\rho\nu} = \varepsilon g(\varepsilon)^{-3/2} (\theta_{\mu\rho} \delta^{\rho\nu} + \varepsilon^2 \theta_{\mu\rho} \tilde{\eta}^{\rho\nu})$$

which determines respectively the Gauss curvature K and the mean curvature M of the surface Σ_ε :

$$\begin{aligned} K(\varepsilon) &= \varepsilon^2 g(\varepsilon)^{-2} k_0 && \text{with } k_0 := \det(\theta_{\mu\nu}) = f_{,11} f_{,22} - f_{,12}^2 \\ M(\varepsilon) &= \varepsilon g(\varepsilon)^{-3/2} (m_0 + \varepsilon^2 m_1) && \text{with } m_0 := \frac{1}{2} \operatorname{tr}(\theta_{\mu\nu}) = \frac{1}{2} (f_{,11} + f_{,22}) \\ \text{and } m_1 &:= \frac{1}{2} \operatorname{tr}(\theta_{\mu\rho} \tilde{\eta}^{\rho\nu}) = \frac{1}{2} (f_{,1}^2 f_{,22} + f_{,2}^2 f_{,11} - 2 f_{,1} f_{,2} f_{,12}). \end{aligned} \tag{4.3}$$

Since we are interested in the case when Σ_ε is asymptotically planar [DEK2], i.e. when $K, M \rightarrow 0$ as $|x| \rightarrow \infty$, we assume (d2).

It is clear from the definition (2.2) that the metric tensor of the layer Ω_ε (as a submanifold of \mathbb{R}^3) has the block form

$$(G_{ij}) = \begin{pmatrix} (G_{\mu\nu}) & 0 \\ 0 & 1 \end{pmatrix} \quad \text{with } G_{\nu\mu} = (\delta_\nu^\sigma - u h_\nu^\sigma) (\delta_\sigma^\rho - u h_\sigma^\rho) g_{\rho\mu} \tag{4.4}$$

and thus $G := \det(G_{ij}) = g(1 - 2Mu + Ku^2)^2$. Following [DEK2], one should make sure that the layer mapping \mathcal{L} is a diffeomorphism. However, this is automatically fulfilled for an arbitrary a provided ε is small enough and the layer is built over surfaces of the special form (2.1). At the same time, $G_{\mu\nu}$ can be estimated by the surface metric:

$$C_- g_{\mu\nu} \leq G_{\mu\nu} \leq C_+ g_{\mu\nu} \quad \text{with } C_\pm := (1 \pm a \rho_m^{-1})^2 \tag{4.5}$$

where $\rho_m^{-1} := \max\{\|k_1\|_\infty, \|k_2\|_\infty\} = \mathcal{O}(\varepsilon)$ and k_1, k_2 are the principal curvatures of Σ_ε , i.e. the eigenvalues of h_μ^ν .

4.2. The Hamiltonian

In the beginning we have identified the particle Hamiltonian with $-\Delta_{\mathbb{D}^\varepsilon}$. We want to replace it by a Schrödinger-type operator which would allow us to employ the result of the previous section. This is achieved by means of the unitary transformation:

$$U : L^2(\Omega_\varepsilon) \rightarrow L^2(\Omega_0, g^{1/2} dx du) : \{\psi \mapsto U\psi := (1 - 2Mu + Ku^2)^{1/2} \psi \circ \mathcal{L}\}$$

which leads to the unitarily equivalent operator:

$$\begin{aligned} H &:= U(-\Delta_{\mathbb{D}^\varepsilon})U^{-1} = -g^{-1/2} \partial_i g^{1/2} G^{ij} \partial_j + V \\ V &= g^{-1/2} (g^{1/2} G^{ij} J_{,j})_{,i} + J_{,i} G^{ij} J_{,j} \quad \text{with } J := \ln \sqrt{1 - 2Mu + Ku^2}. \end{aligned}$$

It makes sense since we suppose that the surface is C^4 -smooth. Using the block form (4.4) of G_{ij} , we can split H into a sum, $H = H_1 + H_2$. For the first operator which is given by the part of H where one sums over the Greek indices (referring to the longitudinal coordinates), owing to (4.5) we have the estimate

$$(C_- / C_+^2)(-\Delta_g + v_1) \leq H_1 \leq (C_+ / C_-^2)(-\Delta_g + v_1) \tag{4.6}$$

which holds in $L^2(\mathbb{R}^2 \times I, g^{1/2} dx du)$ in the form sense. Here $-\Delta_g$ denotes the surface Laplace–Beltrami operator which can be written in the component form as $-g^{-1/2} \partial_\mu g^{1/2} g^{\mu\nu} \partial_\nu$, and

$$v_1 := -\frac{|u^2 \nabla_g K - 2u \nabla_g M|_g^2}{4(1 - 2Mu + Ku^2)^2} + \frac{u^2 \Delta_g K - 2u \Delta_g M}{2(1 - 2Mu + Ku^2)} \tag{4.7}$$

where $|\cdot|_g$ and ∇_g mean, respectively, the norm and the gradient operator induced by the metric $g_{\mu\nu}$. On the other hand,

$$H_2 = -\partial_3^2 + V_2 \quad V_2 = \frac{K - M^2}{(1 - 2Mu + Ku^2)^2} \tag{4.8}$$

where the potential V_2 is attractive because $K - M^2$ can be rewritten by means of the principal curvatures as $-\frac{1}{4}(k_1 - k_2)^2$.

In the second step one uses the inequalities (4.2), which are a consequence of (d1), in order to get the bounds

$$(c_-^2/c_+^3)(-\Delta) \leq -\Delta_g \leq (c_+^2/c_-^3)(-\Delta) \quad \text{in } L^2(\mathbb{R}^2, g^{1/2} dx).$$

It remains to realize that—as another consequence of the uniform ellipticity of the metric—one can identify $L^2(\mathbb{R}^2, g^{1/2} dx) = L^2(\mathbb{R}^2)$, and also $L^2(\mathbb{R}^2 \times I, g^{1/2} dx du) = L^2(\mathbb{R}^2 \times I)$ as sets. If we rescale the longitudinal variable by means of $x \mapsto \sigma_\pm x$ with $\sigma_\pm^2 := c_\mp^3 C_\mp^2 / (c_\pm^2 C_\pm)$, we obtain finally the above indicated bounds:

$$H_- \leq H \leq H_+ \quad \text{with } H_\pm := -\Delta - \partial_3^2 + \varepsilon V_\pm \tag{4.9}$$

where

$$V_\pm(x, u) := \frac{1}{\varepsilon} \left(\frac{C_\pm}{C_\mp^2} v_1 + V_2 \right) (x/\sigma_\pm, u).$$

We notice that, since v_1 and V_2 are ε -dependent—cf (4.3) and the explicit formulae for the potentials (4.7) and (4.8)—the potentials V_\pm are well defined for $\varepsilon = 0$.

4.3. The ground state asymptotics

Suppose that each of the operators H_\pm has for all sufficiently small ε just one eigenvalue $E_\pm(\varepsilon)$. Since H_- is below bounded, the minimax principle tells us that the same is true for H and

$$E_-(\varepsilon) \leq E(\varepsilon) \leq E_+(\varepsilon). \tag{4.10}$$

If we identify ε with λ of the previous section, we see that, under our assumptions, the operators H_\pm are well suited for the application of theorem 3.4.

Indeed, the decay requirements (d1–4) yield that $K, M, |\nabla_g K|_g, |\nabla_g M|_g$ and $\Delta_g K$ tend to zero at infinity, and consequently $(V_\pm)_{11} \rightarrow 0$ as $|x| \rightarrow \infty$. Notice that we do not assume a decay of $\Delta_g M$ at infinity which would require $f_{,\mu\nu\rho\sigma} \rightarrow 0$ as $|x| \rightarrow \infty$. The reason is that the term $u \Delta_g M$ itself in the potential (4.7) vanishes when projected onto the first transverse mode χ_1 . One can thus take $\psi_n(x, u) := \varphi_n(x) \chi_1(u)$, where φ_n with $\|\varphi_n\| = 1$ is a sequence of functions with increasing support which move towards infinity as $n \rightarrow \infty$. If φ_n is properly chosen such ψ_n form a Weyl sequence for H_\pm and any $\kappa_1^2 + \alpha$ with $\alpha \in \mathbb{R}^+$. Using then a Neumann bracketing argument, as mentioned in section 2, in order to show that no $\kappa_1^2 + \alpha$ with $\alpha < 0$ can belong to the essential spectrum, we conclude that $\sigma_{\text{ess}}(H_\pm) = [\kappa_1^2, \infty)$. This verifies (a0) of theorem 3.4. Furthermore, using this result together with (4.9) and the minimax principle, we infer that $\inf \sigma_{\text{ess}}(H) = \kappa_1^2$.

The requirement (a1) holds trivially true because V_\pm are essentially bounded due to the surface regularity assumption and (d1–4). Under (r1–3), explicit expressions for the covariant derivatives of K and M in terms of the partial derivatives of the function f show that even $\sup\{V_\pm(\cdot, u) \mid u \in I\}$ belongs to $L^1(\mathbb{R}^2, (1 + |x|^\delta) dx)$, which gives (a3). Let us remark that we require the L^1 integrability in (r3), which is a stronger condition than the L^2 integrability required for the second and third derivatives of f . The assumption (a2) is a consequence of (a3) and the boundedness of V_\pm .

On the other hand, there is a difference between the Hamiltonian (3.1) and our operators H_\pm because the potentials V_\pm themselves depend on ε . Inspecting the proof of theorem 3.4 we see that one can modify it to get the expansion (3.7) using $\varepsilon \mapsto V_\pm(\varepsilon)$ as well. However, expanding the latter in terms of ε , the final expansion resulting from (3.7) changes, and to ensure that it represents an eigenvalue it is necessary that the lowest-order term in this expansion is not

only nonpositive but rather strictly negative. Recall that one can find (physically interesting) examples of situations when the potential depends on the coupling parameter in a nonlinear way and a bound state in the critical zero-mean case may or may not exist [BCEZ].

Hence one has to examine carefully the behaviour with respect to powers of ε in the expansion obtained from (3.7) by an appropriate change of the integration variables:

$$\begin{aligned} w_{\pm}(\varepsilon) &= \frac{\varepsilon}{2\pi} \sigma_{\pm}^2 \int_{\mathbb{R}^2} (V_{\pm})_{11}(\sigma_{\pm}x; \varepsilon) dx \\ &\quad + \left(\frac{\varepsilon}{2\pi}\right)^2 \sigma_{\pm}^4 \left\{ \int_{\mathbb{R}^2 \times \mathbb{R}^2} (V_{\pm})_{11}(\sigma_{\pm}x; \varepsilon) \left(\gamma_E + \ln \frac{\sigma_{\pm}|x-x'|}{2} \right) \right. \\ &\quad \times (V_{\pm})_{11}(\sigma_{\pm}x'; \varepsilon) dx dx' \\ &\quad - \sum_{j=2}^{\infty} \int_{\mathbb{R}^2 \times \mathbb{R}^2} (V_{\pm})_{1j}(\sigma_{\pm}x; \varepsilon) K_0(k_j(\kappa_1)\sigma_{\pm}|x-x'|) \\ &\quad \left. \times (V_{\pm})_{j1}(\sigma_{\pm}x'; \varepsilon) dx dx' \right\} + \mathcal{O}(\varepsilon^{2+\nu}) \end{aligned}$$

which determines the possible eigenvalues of H_{\pm} via $E_{\pm} = \kappa_1^2 - \exp(2w_{\pm}^{-1})$. By virtue of (4.3) we can write

$$\varepsilon^{-1}V_2(x, u; \varepsilon) = \varepsilon(k_0(x) - m_0(x)^2) + r_2(x, u; \varepsilon)$$

where r_2 is an integrable function collecting the terms of order ε^2 and higher. Hence

$$\forall j \in \mathbb{N} : \quad \varepsilon^{-1}(V_2)_{1j}(x; \varepsilon) = [\varepsilon(k_0(x) - m_0(x)^2) + \mathcal{O}(\varepsilon^2)]\delta_{1j}$$

where we abuse the notation a little because the error term depends on x as well. The expansion of $\varepsilon^{-1}v_1$ requires more attention because $\varepsilon^{-1}\Delta_g M$ in the second term of (4.7) is of order one. First of all, we can write

$$v_1(x, u; \varepsilon) = -u(\Delta_g M)(x; \varepsilon) + u^2 \left(\frac{1}{2} \Delta_g K - |\nabla_g M|_g^2 - 2M \Delta_g M \right)(x; \varepsilon) + r_1(x, u; \varepsilon)$$

where r_1 is an integrable function of order $\mathcal{O}(\varepsilon^3)$. This does not mean, of course, that the first two terms constitute a quadratic polynomial in ε , because $|\cdot|_g$, ∇_g and Δ_g expand as well. Since the first term in the above expansion of v_1 is an odd function of u , it does not contribute to $(V_{\pm})_{11}$. On the other hand, it plays an important role in the higher modes:

$$\begin{aligned} (v_1)_{11}(x; \varepsilon) &= \|u\chi_1\|_I^2 \left(\frac{1}{2} \Delta_g K - |\nabla_g M|_g^2 - 2M \Delta_g M \right)(x; \varepsilon) + \mathcal{O}(\varepsilon^3) \\ (v_1)_{1j}(x; \varepsilon) &= -(\chi_1, u\chi_j)_I (\Delta_g M)(x; \varepsilon) + \mathcal{O}(\varepsilon^2) \quad \text{if } j \in \mathbb{N} \setminus \{1\}. \end{aligned}$$

Notice that the inner product in the second line is nonzero for even j only. Now we use the relations

$$|\nabla_g M|_g^2 = |\nabla M|^2 + \varepsilon^2 M_{,\mu} \tilde{\eta}^{\mu\nu} M_{,\nu} \quad \text{and} \quad -\Delta_g = -\Delta + \varepsilon^2 L(\varepsilon)$$

where $L(\varepsilon)$ is a second-order differential operator with coefficients which expand as $\mathcal{O}(1)$. This, together with (4.3), yields

$$\begin{aligned} \varepsilon^{-1}(v_1)_{11}(x; \varepsilon) &= \varepsilon \|u\chi_1\|_I^2 \left(\frac{1}{2} \Delta k_0 - |\nabla m_0|^2 - 2m_0 \Delta m_0 \right)(x) + \mathcal{O}(\varepsilon^2) \\ \varepsilon^{-1}(v_1)_{1j}(x; \varepsilon) &= -(\chi_1, u\chi_j)_I (\Delta m_0)(x) + \mathcal{O}(\varepsilon). \end{aligned}$$

Since C_{\pm}/C_{\mp}^2 and σ_{\pm} expand like $1 + \mathcal{O}(\varepsilon)$, we arrive at

$$w_{\pm}(\varepsilon) = \frac{\varepsilon^2}{2\pi} \left\{ \int_{\mathbb{R}^2} (k_0 - m_0^2)(x) dx + u\chi_1\|_I^2 \int_{\mathbb{R}^2} \left(\frac{1}{2} \Delta k_0 - |\nabla m_0|^2 - 2m_0 \Delta m_0 \right)(x) dx \right.$$

$$\begin{aligned}
 & -\frac{1}{2\pi} \sum_{j=2}^{\infty} (\chi_1, u\chi_j)_I^2 \int_{\mathbb{R}^2 \times \mathbb{R}^2} (\Delta m_0)(x) K_0(k_j(\kappa_1)\sigma_{\pm}|x-x'|) \\
 & \times (\Delta m_0)(x') dx dx' \Big\} + \mathcal{O}(\varepsilon^{2+\gamma}) \tag{4.11}
 \end{aligned}$$

where in fact the sum runs over even j only.

This expression can be further simplified. By a double integration by parts, where one uses the fact that $f_{,\mu} f_{,\nu\rho} \rightarrow 0$ as $|x| \rightarrow \infty$ due to (d1) and (d2), we get that the integral of k_0 is equal to zero. We note in this connection that it follows then from (4.1) and (4.3) that the total Gauss curvature, $\mathcal{K} := \int_{\Sigma_\varepsilon} K d\Sigma_\varepsilon$, behaves as $\mathcal{O}(\varepsilon^4)$. Using the divergence theorem and the assumptions (d1), (d2) and (d3), by which $\nabla K \rightarrow 0$ as $|x| \rightarrow \infty$, we find that the integral of Δk_0 does not contribute as well. A similar argument employing the Green formula gives

$$\int_{\mathbb{R}^2} (m_0 \Delta m_0)(x) dx = - \int_{\mathbb{R}^2} |\nabla m_0|^2(x) dx. \tag{4.12}$$

It is convenient to rewrite the integral part of the last term in (4.11) by means of a convolution:

$$\frac{1}{2\pi} \int_{\mathbb{R}^2 \times \mathbb{R}^2} (\Delta m_0)(x) K_0(k_j(\kappa_1)\sigma_{\pm}|x-x'|) (\Delta m_0)(x') dx dx' = (\Delta m_0, G_k * \Delta m_0)$$

where $G_k(\cdot) := (2\pi)^{-1} K_0(k|\cdot|)$ and k abbreviates $k_j(\kappa_1)\sigma_{\pm}$. Since G_k is the fundamental solution of the distributional equation $(-\Delta + k^2)G_k = \delta$, we get

$$\begin{aligned}
 (\Delta m_0, G_k * \Delta m_0) &= (\Delta m_0, \Delta G_k * m_0) \\
 &= (\Delta m_0, (k^2 G_k - \delta) * m_0) \\
 &= k^2 (\Delta m_0, G_k * m_0) - (\Delta m_0, m_0) \\
 &= k^2 (G_k * \Delta m_0, m_0) + \|\nabla m_0\|^2 \\
 &= k^2 (\Delta G_k * m_0, m_0) + \|\nabla m_0\|^2 \\
 &= k^2 ((k^2 G_k - \delta) * m_0, m_0) + \|\nabla m_0\|^2 \\
 &= k^4 (m_0, G_k * m_0) - k^2 \|m_0\|^2 + \|\nabla m_0\|^2. \tag{4.13}
 \end{aligned}$$

In the second line we have employed the identity (4.12). If we insert the obtained expression into the expansion (4.11) and use, in addition, the Parseval identity:

$$\|u\chi_1\|_I^2 = \sum_{j=1}^{\infty} (\chi_1, u\chi_j)_I^2 \tag{4.14}$$

we find that the terms containing $\|\nabla m_0\|$ cancel. Hence we arrive at

$$w_{\pm}(\varepsilon) = -\frac{\varepsilon^2}{2\pi} \left\{ \|m_0\|^2 + \sum_{j=2}^{\infty} (\chi_1, u\chi_j)_I^2 k^2 [k^2 (m_0, G_k * m_0) - \|m_0\|^2] \right\} + \mathcal{O}(\varepsilon^{2+\gamma})$$

where, of course, k expands as $k_j(\kappa_1) + \mathcal{O}(\varepsilon)$. Since

$$\sum_{j=2}^{\infty} (\chi_1, u\chi_j)_I^2 k_j(\kappa_1)^2 = \|u \partial_3 \chi_1\|_I^2 - \kappa_1^2 \|u\chi_1\|_I^2 = 1$$

we see also that the terms containing $\|m_0\|$ cancel. Using finally the Fourier transformation as in (3.8), we obtain

$$w_{\pm}(\varepsilon) = -\varepsilon^2 \sum_{j=2}^{\infty} (\chi_1, u\chi_j)_I^2 k_j(\kappa_1)^4 \int_{\mathbb{R}^2} \frac{|\hat{m}_0(\omega)|^2}{|\omega|^2 + \kappa_j^2 - \kappa_1^2} d\omega + \mathcal{O}(\varepsilon^{2+\gamma})$$

where we have also expanded the remaining k from G_k w.r.t. ε and included the higher orders into the error term.

We note that m_0 (and therefore \hat{m}_0) cannot be identically zero once Σ_1 is not a plane. It can be seen from the formulae (4.3): suppose that $m_0 \equiv 0$. Then $f_{,11} \equiv -f_{,22}$, which yields $k_0 = -(f_{,11}^2 + f_{,12}^2)$ and consequently $k_0 \equiv 0$ because we have shown that $\int_{\mathbb{R}^2} k_0(x) dx = 0$. Hence $K \equiv 0$. At the same time, $f_{,\mu\nu} \equiv 0$, which gives $m_1 \equiv 0$ and therefore $M \equiv 0$. If we thus exclude the trivial planar case, the lowest-order term in the expansion is strictly negative. Since it is identical for both w_+ and w_- , it follows by (4.10) that it is the same also for the expansion of $w(\varepsilon)$ in the ground-state energy $E = \kappa_1^2 - e^{2w^{-1}}$ of the Hamiltonian H . This concludes the proof of theorem 2.1.

Remark. The surface Σ_1 is not planar if the matrix $\theta_{\mu\nu}$ is not identically zero, i.e. if $f_{,\mu\nu} \not\equiv 0$. This can be seen as follows. Suppose that $f_{,\mu\nu} = 0$ identically. Solving this as a system of differential equations, we get $f(x) = c_\mu x^\mu$, where c_μ is a constant vector, which is exactly a plane parametrization.

4.4. The thin layer limit

We say that the layer Ω_ε is thin if $a \ll \rho_m$. Since the minimum curvature radius introduced in (4.5) explodes as $\varepsilon \rightarrow 0$ this condition is eventually always satisfied. It is useful, however, to consider also a ‘true’ thin-layer limit in which we start with a mildly curved layer with a fixed ε which is already in the asymptotic regime, and make its width d small. To this aim we rewrite the formula for w_1 in the expansion $w(\varepsilon) = \varepsilon^2 w_1 + \mathcal{O}(\varepsilon^{2+\nu})$ of theorem 2.1 in a different way. For this we go back to the intermediate expansion (4.11). We simplify it as above. However, we do not use (4.13) and (4.14). Instead we apply the transformation (3.8) directly to $(\Delta m_0, G_k * \Delta m_0)$, expand k in terms of ε , thus obtaining

$$w_1 = -\frac{1}{2\pi} \|m_0\|^2 + \frac{\|u\chi_1\|_I^2}{2\pi} \|\nabla m_0\|^2 - \sum_{j=2}^{\infty} (\chi_1, u\chi_j)_I^2 \int_{\mathbb{R}^2} \frac{|\widehat{\Delta m_0}(\omega)|^2}{|\omega|^2 + k_j(\kappa_1)^2} d\omega.$$

An explicit calculation yields $\|u\chi_1\|_I^2 = (\pi^2 - 6)/(12\kappa_1^2)$, so the second term in the expression for w_1 is proportional to d^2 by the definition of κ_1 . On the other hand, using the estimate $|\omega|^2 + k_j(\kappa_1)^2 \geq k_j(\kappa_1)^2 \geq \kappa_2^2 - \kappa_1^2 = 3\kappa_1^2$, together with (4.14) and the above explicit result, we see that the third term is $\mathcal{O}(d^4)$. Note that this makes sense because $\|\widehat{\Delta m_0}\| = \|\Delta m_0\|$ is finite due to (r3). Summing up the argument, we obtain the expansion (2.4).

This formula has a transparent structure. By virtue of (4.3), (4.1) and the fact that $\int_{\mathbb{R}^2} k_0(x) dx = 0$, we get the behaviour

$$\int_{\Sigma_\varepsilon} (K - M^2) d\Sigma_\varepsilon = -\varepsilon(\|m_0\|^2 + \mathcal{O}(\varepsilon^2)).$$

Hence the first term of (2.4) comes from the surface attractive potential $K - M^2$ which dominates the picture for thin layers. The last claim follows from (4.7) and (4.8), because $v_1 \rightarrow 0$ and the leading term of V_2 is $K - M^2$ in the limit $u \rightarrow 0$, which one can do as soon as $d \rightarrow 0$. At the same time, $C_\pm \rightarrow 1$, which in view of (4.6), leads to the ‘limiting’ Hamiltonian $-\Delta_g - \partial_3^2 + K - M^2$. Such a limit is, of course, only formal since the transverse spectrum explodes, which classically corresponds to increasingly rapid oscillations in the transverse direction. Nevertheless, a similar limit with the confinement realized by a harmonic potential transverse to a compact curved surface was discussed recently in a rigorous way in [FH] and led to the same tangential operator $-\Delta_g + K - M^2$.

Acknowledgments

The authors wish to thank to Pierre Duclos for helpful discussions and for the hospitality at Centre de Physique Théorique, Marseille-Luminy, where a part of the work was performed. The research has been partially supported by Grant GAAS A1048101.

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